Journal of Applied Mechanics and Technical Physics, Vol. 47, No. 2, pp. 183–189, 2006

VIBRATIONAL CONVECTION IN THE HELE-SHAW CELL. THEORY AND EXPERIMENT

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UDC 532.5, 532.5.013.4, 532.517.3

The influence of high-frequency horizontal vibrations on convection in the Hele-Shaw cell located in a uniform gravity field is considered experimentally and theoretically. Nonlinear regimes of vibrational convection in the supercritical region are examined. It is shown that horizontal vibrations (directed toward the wide sides of the cell) decrease the threshold of quasi-equilibrium stability. Regions of existence of one- and two-vortex steady flows are found, and unsteady regular and random regimes of thermal vibrational convection are considered. New random regimes in the Hele-Shaw cell are found, which result from nonlinear interaction of the "lower" modes responsible for the formation of regular supercritical convective regimes.

Key words: *thermal vibrational convection, Hele-Shaw cell, regular and random flows.*

Introduction. Convective flows in liquids and gases, which arise under conditions of spatial inhomogeneity of density in the gravity field, are wide-spread types of motion of liquids and gases in nature. Such flows are characterized by a wider range of structures, as compared to isothermal types of motion, whose role is taken into account in design of various engineering devices. Therefore, of significant interest is to consider conditions of origination of gravitational-convective flows and to study their stability and evolution in space and time in various situations, e.g., under the action of variable inertial acceleration, forced flow, porous media, nonuniform composition, magnetic field, or other complicating factors.

Convective motion in a vertical Hele-Shaw cell under the action of high-frequency vibrations is considered in the present work both experimentally and theoretically. The Hele-Shaw cell is a cavity in the form of a rectangular parallelepiped with two linear scales much greater than the third one. The cavity is heated from below and is subjected to horizontal vibrations aligned with the wide sides of the cell. A typical feature of convective motion in the Hele-Shaw cell is that the planes of trajectories are aligned parallel to the wide edges in a wide range of angles of inclination of the cavity. Because of significant thermal and hydraulic resistance, the range of vibrational regimes in the Hele-Shaw cell is much closer to the basic level of instability than in a horizontal layer. For this reason, the Hele-Shaw cell is a convenient object for experimental (first of all, optical) methods of research and for theoretical calculations because the approximation of plane trajectories can be used. Free thermal convection in the Hele-Shaw cell was considered experimentally and theoretically in [1, 2]. The boundaries of equilibrium stability were determined, the structure of steady flows was studied, and regular and some random vibrational regimes were described. It should be noted that the case with these vibrations was one of the first physical examples of the random behavior in simple hydrodynamic systems.

The first theoretical study of vibrational convection in the Hele-Shaw cell in microgravity was performed by Braverman [3] who showed that vibrations induce convection even if there is no force of gravity. From the viewpoint of experimental implementation, of interest are the works [4, 5], where convection was considered in a static field of gravity under the action of linear vibrations; it was shown there that the critical gravitational Rayleigh number decreases with increasing vibrational Rayleigh number. Various regimes of vibrational convection were calculated

Perm' State University, Perm' 614990; babushkin@imail.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 47, No. 2, pp. 40–48, March–April, 2006. Original article submitted December 9, 2004; revision submitted June 16, 2005.

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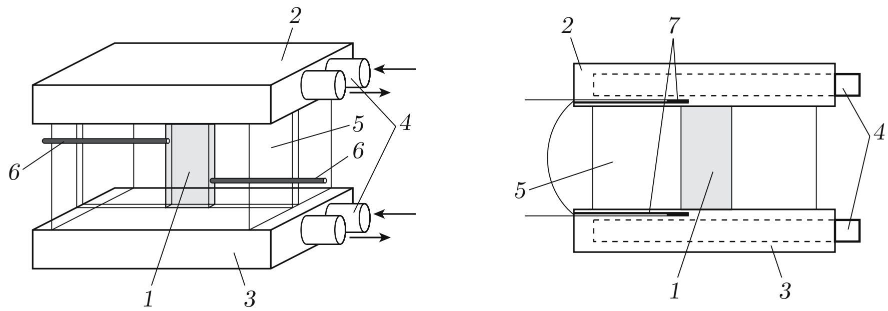

Fig. 1. Convective chamber: working cavity (1), aluminum heat exchangers (2 and 3), pipes (4), Plexiglas array (5), pipes (6), and orifices in heat exchangers (7).

in a wide range of the governing parameters, and various scenarios of the transition to chaos were analyzed. The influence of high-frequency vibrations on stability of mechanical equilibrium and regular and random supercritical convective flows in the Hele-Shaw cell, however, was not considered experimentally in earlier papers.

Formulation of the Problem and Experiment. A laboratory model was made to study convective flows in the Hele-Shaw cell experimentally (Fig. 1). The working cavity (height $h = 40$ mm, length $l = 20$ mm, and thickness $2d = 2.0$ mm) is bounded from above and from below by heat exchangers 16 mm thick. There are channels 7.5 mm in diameter drilled in the heat exchangers to organize opposing flows of the thermostatting fluid in the neighboring orifices to ensure uniform temperatures along the upper and lower boundaries of the cavity. Water in the channels was pumped from jet ultrathermostats through pipes (the temperature was sustained constant within 0.05°C). On the sides, the cavity was bounded by a Plexiglas array 80 mm long and 50 mm wide. The large size of the array almost eliminated all external thermal effects.

The working fluid was poured into the cell through pipes with an outer diameter of 2 mm whose ends were equipped with elastic rubber tubes. When the cavity was filled by the fluid, the rubber tubes were compressed at a distance of 50 mm from the pipes and acted as pressure dampers. The difference in temperature between the heat exchangers was measured by a differential copper–Constantan thermocouple (the diameters of the Constantan and copper wire were 0.20 and 0.15 mm, respectively, and the length of junctions was 1 mm). The thermocouple junctions were placed into orifices in the heat exchangers. The thermocouple data were measured by a V7-54/3 voltmeter. The temperature difference was measured within 0.025°C.

To study the influence of linear vibrations on stability of convective flows in the Hele-Shaw cell, we fabricated a vibration table shown schematically in Fig. 2. The vibrator is a crank mechanism, which transforms the rotational motion of the flywheel (which, in turn, is set into rotational motion by a commutator electric engine via an intermediate shaft with a sheave through belt transmission) to back-and-forth motion of the table with the cavity along the guides. The use of sheaves of different diameters on the flywheel shaft and engine (14 and 5 cm, respectively) has a positive effect on accuracy of sustaining the vibration frequency. Rotational motion of the flywheel is converted to translational motion of the carriage by means of a rod attached to the flywheel and pusher through bearings. The use of a commutator engine allows smooth variation of vibration frequency during the experiment. The amplitude of vibrations of the carriage is 6 cm.

The frequency of vibrations is determined by a ChZ-54 frequency meter with pulses of the photon-coupled pair designed on the basis of an integral microcircuit being fed to the frequency-meter input. The light flux from the light-emitting diode on the photodetector is interrupted by a plate fixed on the vibrator carriage. The same scheme is used to simultaneously trigger the flash lamp during visual observations and video filming. The experiments were performed with consecutive variation of the temperature difference between the heat exchangers with a fixed amplitude and frequency of vibrations of the carriage or with a constant amplitude and temperature difference by changing the vibration frequency. For visualization of convective flows, light-scattering particles of aluminum powder, which can be easily entrained by the flow, were added to the working fluid. The reflection factor of these particles strongly depends on their orientation, which permits observation of the motion pattern as a whole and the trajectories of individual particles. As the addition of a small number of particles into the examined fluid changes the parameters of the latter only slightly, their values in calculations were assumed to be equal to those of the pure fluid at an ambient temperature of 20°C.

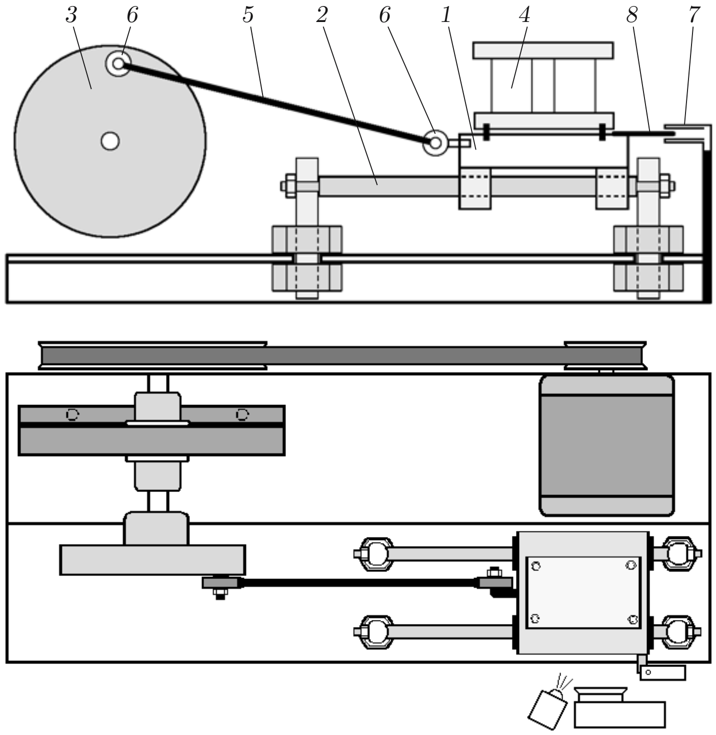

Fig. 2. Experimental setup: 1) table; 2) guides; 3) flywheel; 4) cavity; 5) carriage; 6) bearings; 7) photon-coupled pair; 8) plate.

The flow structure was registered by a digital video camera. The films were divided into individual frames and were processed on a computer by the "Adobe Photoshop 7.0" graphic editor. The images in the frames were overlapped, and the resultant image was an analog of a photograph with a large exposure time, which clearly shows the tracks of moving particles.

The working fluid was transformer oil (an experiment with the same fluid in a cell of the same configuration but without vibrations was performed previously). The time of stabilization of a certain convection regime, which was estimated on the basis of characteristic times of relaxation of temperature and hydrodynamic perturbations, was approximately 1.5 h. The time of reaching a steady regime in the experiment was more than two hours. In the experiment, the maximum temperature difference was 35°C, and the vibration frequency was 9.2 Hz (overloads up to $20g$).

The governing parameters were the Prandtl number $\mathrm{Pr} = \nu/\chi$ and also the thermal and vibrational Rayleigh numbers [5] $\mathrm{Ra} = g\beta A d^4/(\nu\chi)$ and $\mathrm{Ra}_v = (b\Omega\beta A d^2)^2/(2\nu\chi)$, where g is the acceleration of gravity, A is the characteristic temperature gradient, d is the half-thickness of the cavity, β, ν, and χ are the thermal expansion coefficient, kinematic viscosity, and thermal diffusivity, respectively, b is the amplitude, and Ω is the circular frequency of vibrations. The characteristic temperature gradient was determined in terms of the cavity height h and the temperature difference Θ in the upper and lower heat exchangers ($A = \Theta/h$).

Calculation Technique. In accordance with the Hele-Shaw approximation, two linear sizes of the cell should be much greater than the third size. In our case, we assume that $h \gg d$ and $l \gg d$ (l is the cell length). Constraints on the cell thickness allow the use of the approximation of plane trajectories in modeling convection; in this approximation, convective motions in the fluid can occur only in the plane of wide sides (x, y): $\boldsymbol{v} = (v_x, v_y, 0)$ and $\boldsymbol{w}(w_x, w_y, 0)$.

The cavity is subjected to high-frequency vibrations along the unit vector $\boldsymbol{n}$ aligned along the wise sides of the cell. The theoretical description of the averaged convective motion in the Hele-Shaw cell is based on the classical equations of thermal vibrational convection [6] used for numerical and analytical calculations of various regimes of vibrational convection by the Galerkin–Kantorovich method. The system of averaged equations that describe thermal vibrational convection in a nonuniformly heated closed cavity was first obtained in [7] and can be written in the following dimensionless form:

$$\frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{\mathrm{Pr}}\,(\boldsymbol{v}\nabla)\boldsymbol{v} = -\nabla p + \Delta \boldsymbol{v} + \mathrm{Ra}\,T\gamma + \mathrm{Ra}_v(\boldsymbol{w}\nabla)(T\boldsymbol{n} - \boldsymbol{w}); \tag{1}$$

$$\mathrm{Pr}\,\frac{\partial T}{\partial t} + \boldsymbol{v}\nabla T = \Delta T, \qquad \mathrm{div}\,\boldsymbol{v} = 0; \tag{2}$$

$$\mathrm{rot}\,\boldsymbol{w} = \nabla T \times \boldsymbol{n}, \qquad \mathrm{div}\,\boldsymbol{w} = 0. \tag{3}$$

Here $\boldsymbol{v}$, T, and p are averaged fields of velocity, temperature, and pressure changing slowly in time, $\boldsymbol{w}$ is an additional "slow" variable proportional to the amplitude of the fluctuating component of velocity, and γ is a unit vector directed vertically upward. The equations contain three dimensionless parameters: Ra, Ra_v, and Pr.

It is assumed that the value of the parameter Ra_v proportional to the product of the low amplitude of vibrations b and the high frequency Ω is finite. In the general case, an additional parameter of the problem is the ratio of the wide sides l/h, which is assumed to be 1/2 in the present calculations.

The mean component of velocity on the solid boundaries of the cavity is subjected to the adhesion conditions $\boldsymbol{v}|_S = 0$, whereas only the no-slip condition $\boldsymbol{w}_n|_S = 0$ can be imposed onto the amplitude of the fluctuating component of velocity.

Concerning the thermal boundary conditions, we consider the Hele-Shaw cell with thermally insulated vertical faces with a constant temperature gradient corresponding to heating from below.

The geometry of the problem allows us to reduce a three-dimensional problem to a two-dimensional one; therefore, vibrational convection is further analyzed on the basis of equations written in terms of the stream functions and temperature:

$$v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x}; \qquad w_x = \frac{\partial F}{\partial y}, \quad w_y = -\frac{\partial F}{\partial x}.$$

Here $\psi(x, y)$ and $F(x, y)$ are the stream functions for the vector fields $\boldsymbol{v}$ and $\boldsymbol{w}$, respectively. System (1)–(3) in terms of the stream functions has the form

$$\frac{\partial \varphi}{\partial t} + \frac{1}{\mathrm{Pr}}\Big(\frac{\partial \psi}{\partial x}\,\frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial y}\,\frac{\partial \varphi}{\partial x}\Big) = \Delta\varphi - \mathrm{Ra}\,\frac{\partial T}{\partial x} + \mathrm{Ra}_v\Big(\frac{\partial T}{\partial y}\,\frac{\partial^2 F}{\partial x\,\partial y} - \frac{\partial T}{\partial x}\,\frac{\partial^2 F}{\partial y^2}\Big); \tag{4}$$

$$\mathrm{Pr}\,\frac{\partial T}{\partial t} + \frac{\partial \psi}{\partial x}\,\frac{\partial T}{\partial y} - \frac{\partial \psi}{\partial y}\,\frac{\partial T}{\partial x} = \Delta T + \frac{\partial \psi}{\partial x}, \qquad \Delta_1 F = -\frac{\partial T}{\partial y}; \tag{5}$$

$$\varphi = -\Delta_1\psi, \qquad \Delta_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

where φ is the vorticity and Δ_1 is the plane Laplace operator.

In accordance with the Galerkin–Kantorovich procedure, the fields of the stream functions and temperature were decomposed in terms of spatial basis functions with time-dependent amplitudes. The following trigonometric functions were used as a basis:

$$\psi = \sum_{n,m} \psi_{n,m}(t)\sin\Big(\frac{n\pi}{l}\,x\Big)\sin\Big(\frac{m\pi}{h}\,y\Big)\cos\Big(\frac{\pi}{2}\,z\Big),$$

$$T = \sum_{n,m} \theta_{n,m}(t)\cos\Big(\frac{n\pi}{l}\,x\Big)\sin\Big(\frac{m\pi}{h}\,y\Big),$$

$$F = \sum_{n,m} f_{n,m}(t)\cos\Big(\frac{n\pi}{l}\,x\Big)\cos\Big(\frac{m\pi}{h}\,y\Big).$$

The fact that the basis functions for temperature are independent of the transverse coordinate z is consistent with the condition of thermal insulation of the wide faces. In the case of the stream function for the amplitude of the fluctuating component of velocity, the independence of z means that only the no-slip condition is imposed on the vector field $\boldsymbol{w}$.

In expanding the stream functions and temperature into series, we take into account the following modes:

$$\psi_{11}, \quad \psi_{12}, \quad \psi_{21}, \quad \psi_{22}, \quad \psi_{31},$$
$$\theta_{01}, \quad \theta_{02}, \quad \theta_{11}, \quad \theta_{12}, \quad \theta_{21}, \quad \theta_{22}, \quad \theta_{31},$$
$$f_{01}, \quad f_{02}, \quad f_{11}, \quad f_{12}, \quad f_{21}, \quad f_{22}, \quad f_{31}.$$

Note that the stream function of the amplitude of the fluctuating component of velocity, like temperature, is expanded in terms of seven basis functions. Poisson's equation for the stream function of the amplitude of the fluctuating component of velocity is not an evolutionary one. Integration of this equation yields a system of algebraic equations, which allows one to express the amplitudes of the stream function of the "fluctuating" field f_{nm} in terms of appropriate amplitude coefficients of the temperature field θ_{nm}. Thus, the "vibrational force" in the Navier–Stokes equation (4) is completely expressed via the amplitudes of the temperature field.

After weighted integration of vibrational convection equations, we obtain a system of 12 ordinary differential equations for amplitudes, which describes the evolution of perturbations in the system. These equations are not cited here because they are too cumbersome.

The solution of the linear problem of stability of mechanical quasi-equilibrium can be obtained analytically. The boundary of stability for different modes indexed by n and m, which are responsible for periodicity of perturbations along the x and y axes, respectively, has the form

$$\mathrm{Ra} = \frac{\pi^6 l^2}{8n^2}\Big(\frac{n^2}{l^2} + \frac{m^2}{h^2}\Big)^2\Big[\frac{n^2}{l^2} + \frac{m^2}{h^2} + \frac{1}{4}\Big] - \mathrm{Ra}_v \, \frac{n^2 h^2}{n^2 h^2 + m^2 l^2}. \tag{6}$$

An analysis of Eq. (6) shows that the most dangerous factor for the Hele-Shaw cell with the ratio of the sides 2 : 20 : 40 is a one-vortex perturbation (mode ψ_{11}). The threshold of stability determined from the linear theory agrees with the results calculated with the use of the "full" nonlinear system of amplitude equations within 3%.

Within the system obtained, we can identify a closed triplet system (one amplitude for the stream function and two for temperature), which differs from the classical Lorentz triplet by the presence of an additional term characterizing the mean vibrational effect. This allows analytical construction of steady-state solutions for various values of the vibrational and gravitational Rayleigh number and the Prandtl number.

The "full" system of equations for the amplitudes (five for the stream functions and seven for temperature) was analyzed numerically. The system was integrated by the Runge–Kutta–Fehlberg method of the fourth or fifth order of accuracy [8] with the step chosen automatically; the programming language was Fortran-90. Thus, we were able to track the evolution of the system for various values of dimensionless parameters: reaching steady values of the fields of the stream function and temperature; regular vibrational and irregular regimes. A time-dependent method was used in the calculations. This allowed us to describe various types of steady flows and vibrational and random regimes of vibrational convection. The Runge–Kutta method was no longer refined when the difference between the maximum of the stream function at a certain step and its value at the previous step reached 1% during establishment of a regular flow.

Discussion of Experimental and Theoretical Results. In most experiments, we considered the scenarios of the transition to chaos, depending on the temperature difference on the upper and lower boundaries of the cavity under the action of vibrations with fixed amplitude and frequency. A theoretical study, which describes this experiment, is the calculation of the sequence of convective regimes depending on the thermal Rayleigh number with a fixed convective Rayleigh number. The experimental and calculated results show that a certain particular scenario of the transition from the basic state to chaos occurs in the Hele-Shaw cell with the ratio of the sides 2 : 20 : 40 for different fixed values of the vibrational Rayleigh number. For low values of the thermal Rayleigh number, the fluid is in the basis state, which is called mechanical quasi-equilibrium. A typical feature of this state is the presence of velocity oscillations with the frequency of the external action in the fluid, though the fluid as a whole is motionless. Note that the state of mechanical quasiequilibrium in the Hele-Shaw cell with the considered ratio of the sides is observed for all values of the vibrational Rayleigh number that can be reached in the experiment. The calculations show (and the experiments confirm) that the threshold of stability of mechanical quasi-equilibrium decreases with increasing vibrational Rayleigh number. With increasing thermal Rayleigh number, quasi-equilibrium loses its stability, and "soft" excitation of a steady one-vortex flow is observed at a certain critical value of this parameter. Figure 3a shows the isolines of the stream functions calculated for $\mathrm{Ra} = 0.2$ and $\mathrm{Ra}_v = 0.1$ (on the left) and the experimental data for $\mathrm{Ra} = 0.9$ and $\mathrm{Ra}_v = 0.01$ (on the right). Unfortunately, it was not possible to

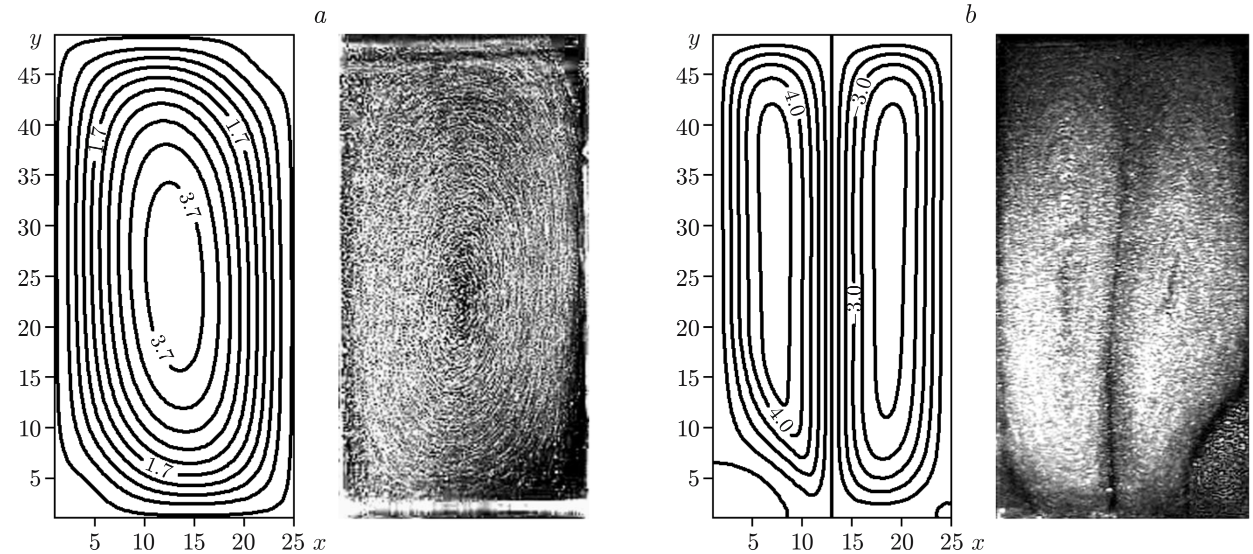

Fig. 3. Steady regimes of vibrational convection: (a) one-vortex flow; (b) two-vortex flow (the calculated and experimental results are shown on the left and on the right, respectively).

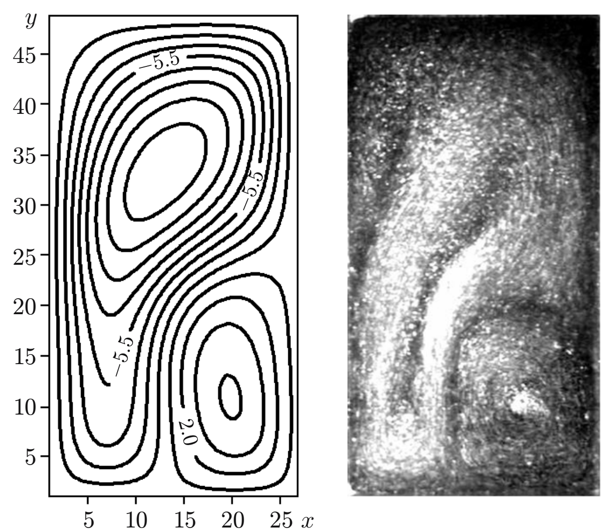

Fig. 4. Vibrational regime at a certain time (the calculated and experimental results are plotted on the left and on the right, respectively).

perform experiments for high values of the vibrational Rayleigh number: the maximum frequencies correspond to $\mathrm{Ra}_v \approx 0.8$. The technique of the theoretical analysis, nevertheless, does not impose any constraints on the values of the dimensionless parameters of the problem. It is shown in calculations that "stiff" excitation of convection is possible in the case of large values of the vibrational Rayleigh number.

With a further increase in the Rayleigh number, the one-vortex flow is replaced by a two-vortex steady regime. Figure 3b shows the calculation results for $\mathrm{Ra} = 0.7$ and $\mathrm{Ra}_v = 0.1$ and the experimental data for $\mathrm{Ra} = 1.66$ and $\mathrm{Ra}_v = 0.01$. To analyze supercritical motions, we constructed the isolines of the averaged fields of the stream function and temperature and also the dependences of the maximum value of the stream function on the vibrational Rayleigh number. The calculations show that a more intense vibrational action on the system with a fixed Rayleigh number in the supercritical region enhances convective motion (thermal vibrational convection phenomenon). A subsequent increase in the Rayleigh number leads to a random vibrational regime, which, in turn, loses its stability and transforms to a regular four-vortex vibrational regime with reconnection of vortices [4, 5].

An example of the random vibrational regime at a certain time is shown in Fig. 4 (the calculated and experimental results are shown on the left and on the right, respectively). A small counter-rotating vortex is incipient in the bottom right corner of the cavity on the background of the one-vortex flow. This vortex increases with time and, interacting with the basic flow, is absorbed by the latter at a certain time. After a certain period, this process is repeated. As a result, vibrations of the lower vortex on the background of the basic one-vortex flow are observed. Visual observations suggest that vibrational flows are random, and the characteristic time of vibrations is 1 to 10 min, depending on the temperature difference.

The experiment and calculations yield different boundaries for the same regimes. The calculated threshold of stability of mechanical quasi-equilibrium and the boundaries of stability of the one-vortex, two-vortex, and vibrational flows and the four-vortex vibrational regime with reconnection of vortices are substantially lower than the experimental values. The reason for this disagreement is the condition of thermal insulation of the wide sides of the cell used in calculations, whereas the wide sides in the experimental model are made of Plexiglas, which corresponds to an intermediate thermal conductivity of the boundaries. It should be noted that the theoretical study [4, 5] performed for ideally heat-conducting wide sides yield a different sequence of convective regimes. The present calculations performed for thermally insulated wide sides offer a more realistic description for the shape and sequence of vibrational convective regimes with increasing governing parameters of the problem.

At high values of the vibrational Rayleigh number, one cannot avoid taking into account the presence of narrow side faces. In the case of a weak vibrational action, the Hele-Shaw cell can be considered as a cavity with infinitely large wide sides. It is only in this case that the state of mechanical quasi-equilibrium can be reached in the cavity. As the vibrational Rayleigh number increases, this approximation becomes invalid. In the calculations, the effect of the narrow sides is assumed to be negligibly small, which automatically imposes certain restrictions on the values of the vibrational Rayleigh number. The validity of the assumption about the infinity of the wide sides of the cavity is supported by the fact of mechanical quasi-equilibrium, which is observed in experiments as the basic state for low values of the thermal Rayleigh number.

Conclusions. Thermal vibrational convection in the Hele-Shaw cell heated from below under the action of horizontal vibrations aligned with the wide sides of the cavity is considered experimentally and theoretically. The ranges of existence of various steady and unsteady vibrational convective regimes are determined. New stable self-oscillatory regimes are found, which are called the vibrational flows. The results of numerical calculations are in agreement with experimental data.

The authors are grateful to G. F. Putin (Perm' State University) for supporting this work and valuable comments.

This work was partly supported by the Russian Foundation for Basic Research (Grant No. 04-02-96026) and by the U.S. Civilian Research and Development Foundation (Grant No. PE-009-0).

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